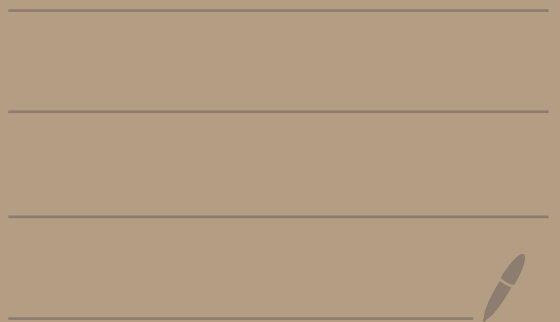


Math 4650

Homework 1

Solutions

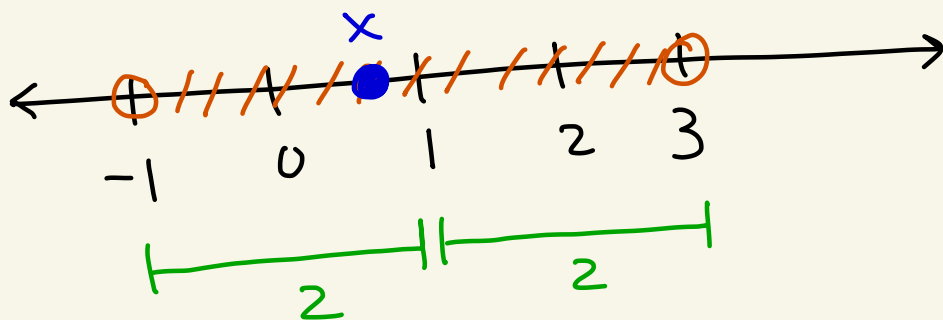


①

(a) $|x-1| < 2$

$$-2 < x-1 < 2$$

$$-1 < x < 3$$

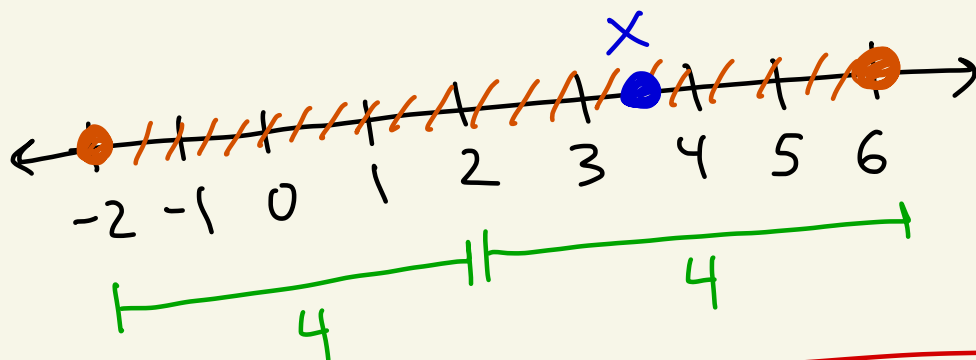


(b) $|-2+x| \leq 4$

$$|x-2| \leq 4$$

$$-4 \leq x-2 \leq 4$$

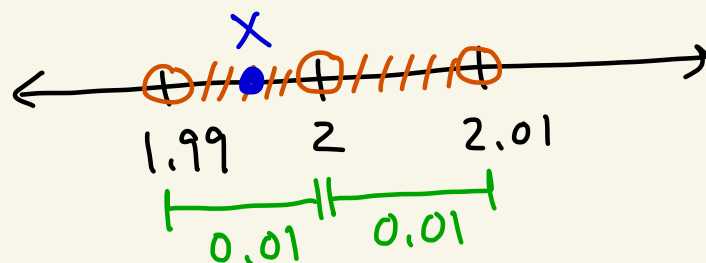
$$-2 \leq x \leq 6$$



(c) $0 < |x-2| < 0.01$

$$-0.01 < x-2 < 0.01 \text{ and } x \neq 2$$

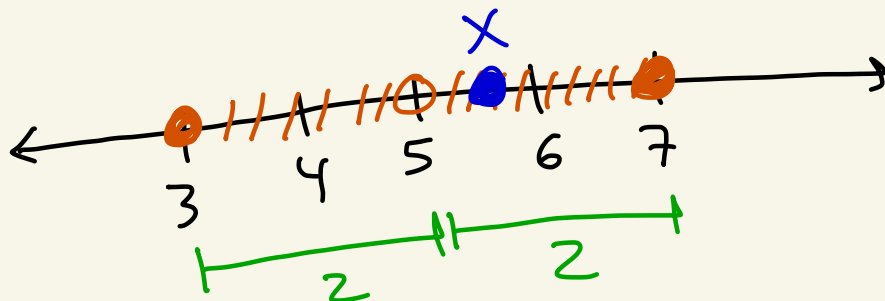
$$1.99 < x < 2.01 \text{ and } x \neq 2$$



(d) $0 < |x-5| \leq 2$

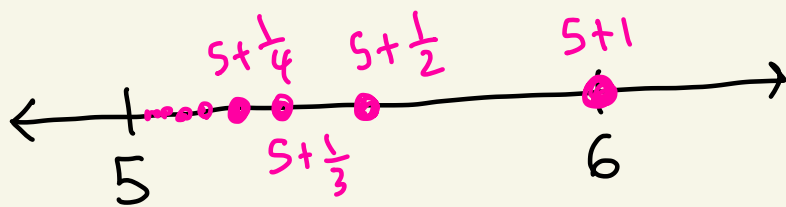
$$-2 \leq x-5 \leq 2 \text{ and } x \neq 5$$

$$3 \leq x \leq 7 \text{ and } x \neq 5$$



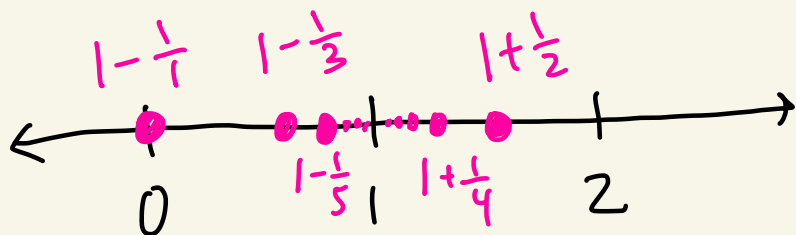
(2)

(a) $X = \{5 + \frac{1}{n} \mid n \in \mathbb{N}\} = \{5 + 1, 5 + \frac{1}{2}, 5 + \frac{1}{3}, 5 + \frac{1}{4}, \dots\}$



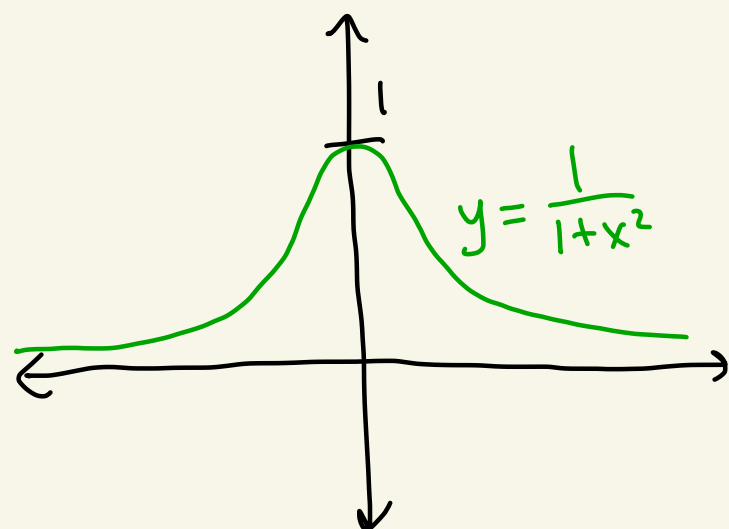
$$\inf(X) = 5$$
$$\sup(X) = 6$$

(b) $X = \{1 + \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$
 $= \{1 - \frac{1}{1}, 1 + \frac{1}{2}, 1 - \frac{1}{3}, 1 + \frac{1}{4}, 1 - \frac{1}{5}, \dots\}$



$$\inf(X) = 0$$
$$\sup(X) = 1 + \frac{1}{2} = 1.5$$

(c) $X = \{\frac{1}{1+x^2} \mid x \in \mathbb{R}\}$

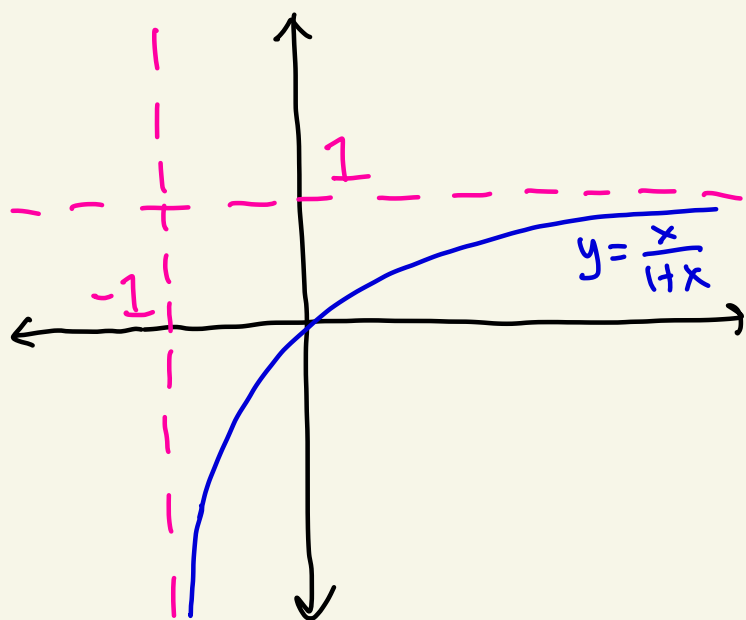


The set X consists of the y -values of this graph.

$$\inf(X) = 0$$

$$\sup(X) = 1$$

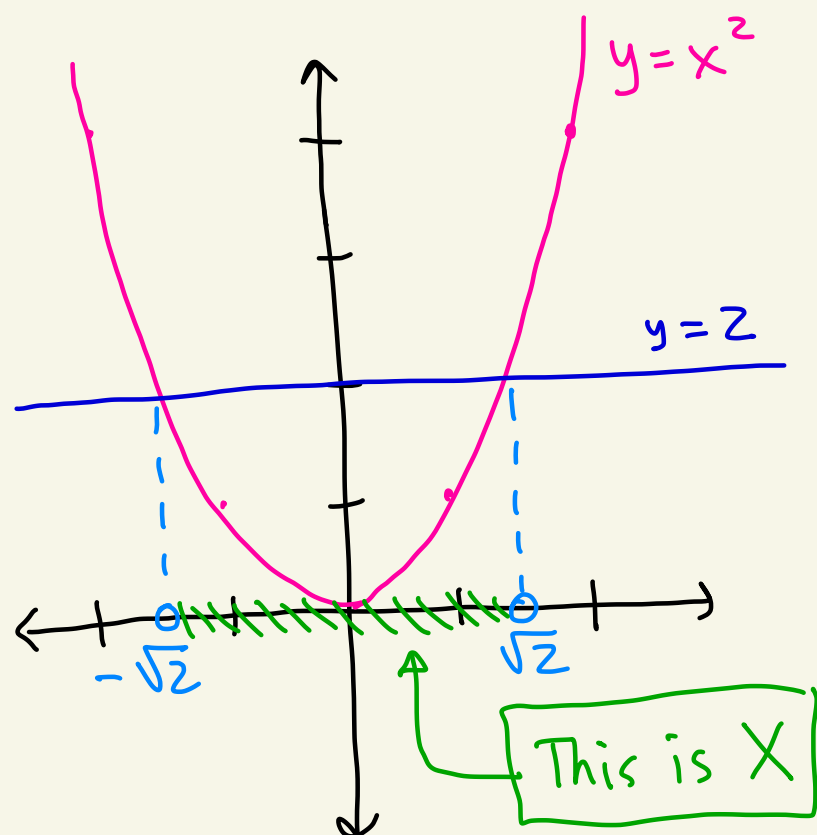
$$(d) X = \left\{ \frac{x}{1+x} \mid x \in \mathbb{R} \text{ with } -1 < x \right\}$$



The set X consists of the y -values of this graph with $-1 < x$
 $\inf(X)$ does not exist
 $\sup(X) = 1$

$$(e) X = \{ x \in \mathbb{R} \mid x^2 + 1 < 3 \}$$

$$= \{ x \in \mathbb{R} \mid x^2 < 2 \}$$



We see from the picture that

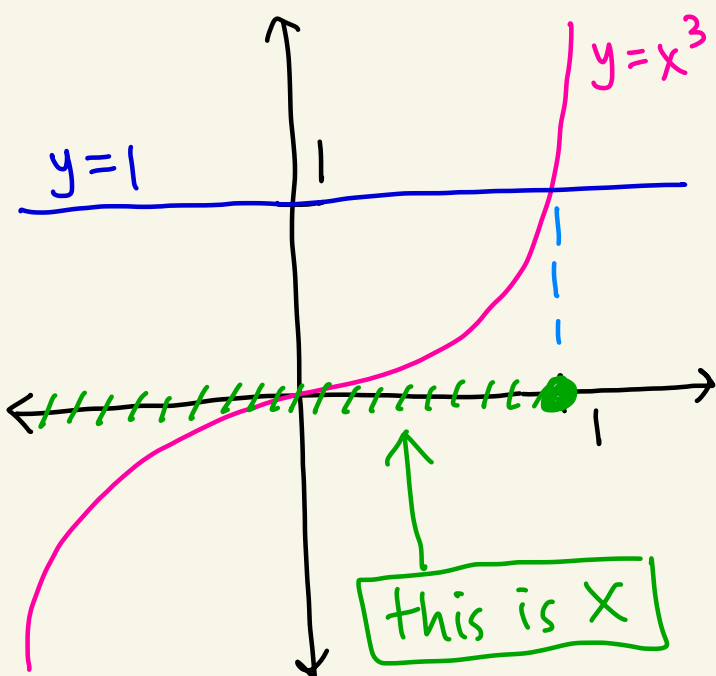
$$X = \{ x \in \mathbb{R} \mid -\sqrt{2} < x < \sqrt{2} \}$$

$$= (-\sqrt{2}, \sqrt{2})$$

$$\inf(X) = -\sqrt{2}$$

$$\sup(X) = \sqrt{2}$$

$$(f) \quad X = \{x \in \mathbb{R} \mid x^3 \leq 1\}$$



From the picture we see that

$$X = \{x \in \mathbb{R} \mid x \leq 1\} \\ = (-\infty, 1]$$

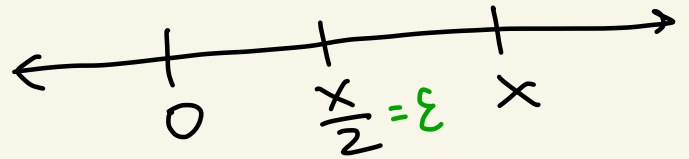
$\inf(X)$ does not exist
 $\sup(X) = 1$

③ Let $x \in \mathbb{R}$ with $x \geq 0$.

Our assumption is that $x \leq \varepsilon$ for all $\varepsilon > 0$.
Let's show this implies that $x = 0$.

Suppose $x > 0$.

Then, $0 < \frac{x}{2} < x$.



Set $\varepsilon = \frac{x}{2}$

By assumption $x \leq \varepsilon$.

But then both $\varepsilon < x$ and $x \leq \varepsilon$
which is a contradiction.

Hence $x > 0$ cannot be true.

So, $x = 0$.



(4)

Suppose that a and b are both supremums for S .

Then a and b are both upper bounds for S .

Since a is a supremum for S and b is an upper bound for S , by def of supremum, we have that $a \leq b$.

a is a least upper bound for S

Since b is a supremum for S and a is an upper bound for S , by def of supremum we have that $b \leq a$.

b is a least upper bound for S

Since $a \leq b$ and $b \leq a$ we have that $a = b$. 

⑤

We are given that b is an upper bound for S and that $b \in S$.

Let's show that $b = \sup(S)$.

(i) We already have that b is an upper bound for S .

(ii) Let's show that b is the least upper bound for S .

Let c be another upper bound for S .

Then, $x \leq c$ for all $x \in S$.

Since $b \in S$ this gives $b \leq c$.

Thus, b is the least upper bound for S .

By (i) and (ii), $b = \sup(S)$.



(6)(a)

Suppose A and B are non-empty subsets of $\mathbb{R}$ bounded from above and below.

Further assume that $A \subseteq B$.

Let $s_A = \sup(A)$ and $s_B = \sup(B)$.

Since s_B is an upper bound for B
we know that $b \leq s_B$ for all $b \in B$.

This implies, because $A \subseteq B$, that
 $a \leq s_B$ for all $a \in A$.

Thus, s_B is an upper bound for A .

Since s_A is the least upper bound
for A we know that $s_A \leq s_B$.

Thus, $\sup(A) \leq \sup(B)$.

$\inf(B) \leq \inf(A)$.

A similar argument shows
You try.

Also, if $a \in A$, by def. we have

$\inf(A) \leq a \leq \sup(A)$.

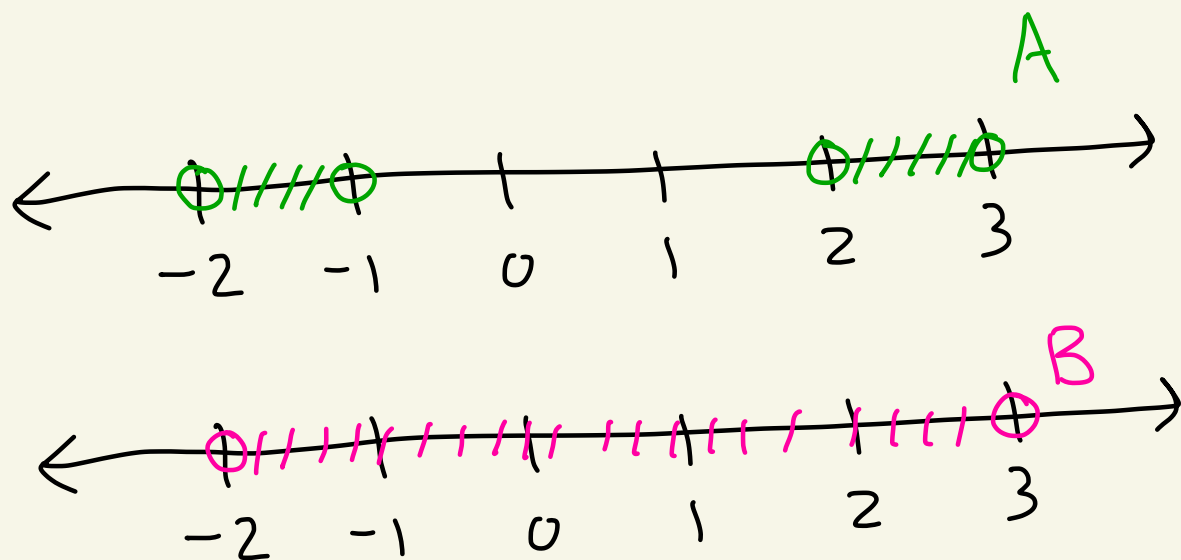
Hence, $\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$



(6)(b)

$$\text{Set } A = (-2, -1) \cup (2, 3)$$

$$B = (-2, 3)$$



Here $\inf(A) = -2 = \inf(B)$ and
 $\sup(A) = 3 = \sup(B)$

but $A \neq B$

7(a) (We give two different proofs)

PROOF #1 - USING DEF OF SUPREMUM

proof: We are given that $s_A = \sup(A)$ and $s_B = \sup(B)$ exist. We are also given that $A \cap B \neq \emptyset$.

Since $A \cap B \subseteq A$ and $s_A = \sup(A)$ we know that $x \leq s_A$ for all $x \in A \cap B$.

Since $A \cap B \subseteq B$ and $s_B = \sup(B)$ we know that $x \leq s_B$ for all $x \in A \cap B$.

Thus, s_A and s_B are upper bounds for $A \cap B$.

Thus, $A \cap B$ is bounded from above and $\sup(A \cap B)$ exists.

Let $m = \min\{s_A, s_B\}$.

Then from above m is an upper bound for $A \cap B$.

By the def of supremum (least upper bound) we know that $\sup(A \cap B) \leq m$.

Thus, $\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$



PROOF #2 - USING THE INF/SUP THEOREM

Proof:

Let $s_A = \sup(A)$.

Since $A \cap B \subseteq A$ we know that $x \leq s_A$ for all $x \in A \cap B$.

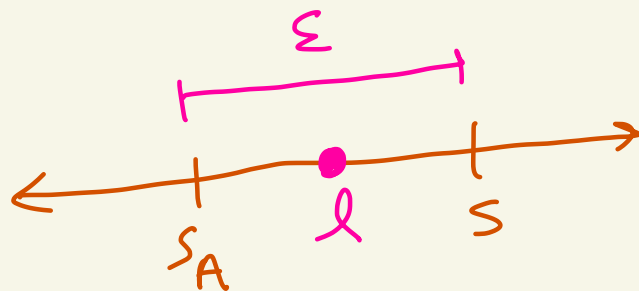
So, $A \cap B$ is bounded from above by s_A .

Thus, $s = \sup(A \cap B)$ exists.

Let's show that $s \leq s_A$.

Suppose that $s > s_A$.

Then, $\varepsilon = s - s_A > 0$.



By the inf/sup theorem

Since $s = \sup(A \cap B)$, there exists $l \in A \cap B$

with $s_A < l \leq s$.

$\underbrace{s_A}_{s-\varepsilon}$

But then $l \in A$ and $s_A < l$ which contradicts the fact that $s_A = \sup(A)$.

Therefore, $s \leq s_A$.

Similarly one can show that $s \leq s_B$.

Thus, $s \leq \min\{s_A, s_B\}$

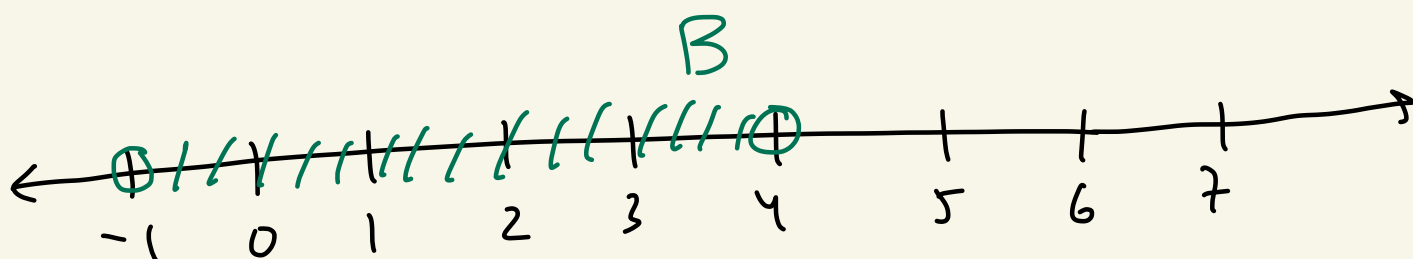
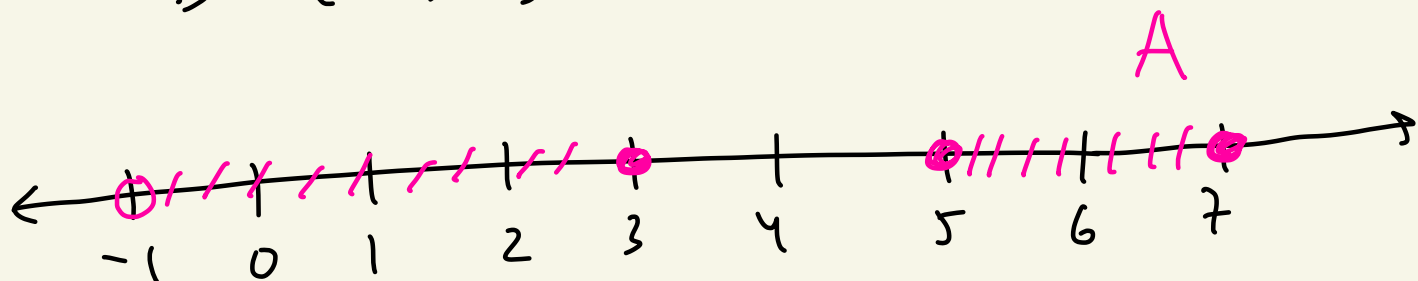
So, $\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$.



(7)(b)

$$\text{Let } A = (-1, 3] \cup [5, 7]$$

$$B = (-1, 4)$$



$$\text{Then, } A \cap B = (-1, 3]$$

$$\text{And } \sup(A) = 7$$

$$\sup(B) = 4$$

$$\min \{ \sup(A), \sup(B) \} = 4$$

$$\sup(A \cap B) = 3$$

Thus,

$$\sup(A \cap B) \neq \min \{ \sup(A), \sup(B) \}$$

(7)(c) (We give two different proofs)

PROOF #1 - USING DEF OF SUPREMUM

proof:

Let $s_A = \sup(A)$ and $s_B = \sup(B)$.

We will assume that $s_A \leq s_B$.

Since $s_A \leq s_B$ we have that

$$s_B = \max \left\{ \underset{\substack{\uparrow \\ s_A}}{\sup(A)}, \underset{\substack{\uparrow \\ s_B}}{\sup(B)} \right\}$$

If $s_B \leq s_A$, the same proof would work with A and B interchanged

We first show that s_B is an upper bound for $A \cup B$.

Let $x \in A \cup B$.

Then $x \in A$ or $x \in B$.

If $x \in A$, then $x \leq s_A \leq s_B$.

If $x \in B$, then $x \leq s_B$

Thus, no matter the case we have $x \leq s_B$.

So, s_B is an upper bound for $A \cup B$.

We will show that s_B is the least upper bound for $A \cup B$.

Suppose c is another upper bound for $A \cup B$.

Then $x \leq c$ for all $x \in A \cup B$.

Thus, $x \leq c$ for all $x \in B$.

So, c is an upper bound for B .

Since s_B is the least upper bound for B
we get that $s_B \leq c$.

Thus, s_B is the least upper bound for $A \cup B$.

That is, $s_B = \sup(A \cup B)$.

$$\text{So, } \sup(A \cup B) = \max \{ \underbrace{\sup(A), \sup(B)}_{s_B} \}$$



PROOF #2 - USING THE INF/SUP THEOREM

proof:

Let $s_A = \sup(A)$ and $s_B = \sup(B)$.

We will assume that $s_A \leq s_B$.

Since $s_A \leq s_B$ we have that

$$s_B = \max \{ \underbrace{\sup(A)}_{s_A}, \underbrace{\sup(B)}_{s_B} \}$$

If $s_B \leq s_A$, the same proof would work with A and B interchanged

We first show that s_B is an upper bound for $A \cup B$.
Let $x \in A \cup B$.

Then $x \in A$ or $x \in B$.

If $x \in A$, then $x \leq s_A \leq s_B$.

If $x \in B$, then $x \leq s_B$.

Thus, no matter the case we have $x \leq s_B$.
So, s_B is an upper bound for $A \cup B$.

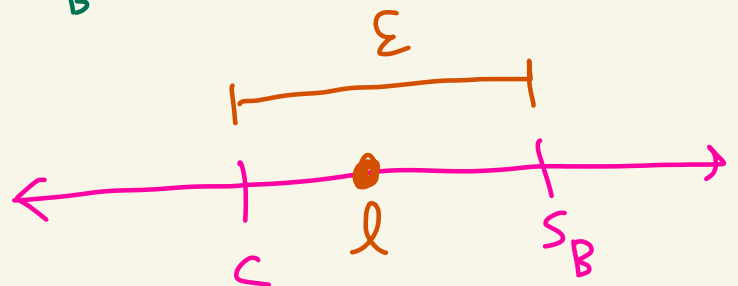
Now we show that s_B is the least upper bound for $A \cup B$.

Suppose that c is another upper bound for $A \cup B$.

We need to show that $s_B \leq c$.

Suppose that $s_B > c$.

Then $\varepsilon = s_B - c > 0$.



By the inf/sup theorem,
since $s_B = \sup(B)$, there exists $l \in B$
with $\underbrace{c < l}_{s_B - \varepsilon} \leq s_B$

Then $l \in A \cup B$ and $c < l$.

This contradicts the fact that c is

an upper bound for $A \cup B$.

Thus, $s_B > c$ can't be true.

So, $s_B \leq c$.

Thus, s_B is the least upper bound
for $A \cup B$.

So, $s_B = \sup(A \cup B)$.

Thus, $\underbrace{\max\{\sup(A), \sup(B)\}}_{s_B} = \sup(A \cup B)$.



⑧(a)

We break the proof into two cases.

Case 1: Suppose $a \leq b$.

Then, $a - b \leq 0$.

$$\text{So, } |a - b| = -(a - b) = b - a$$

Also, $b - a \geq 0$.

$$\text{So, } |b - a| = b - a$$

$$\text{Thus, } |a - b| = |b - a|.$$

Case 2: Suppose $a > b$

Then, $a - b > 0$.

$$\text{So, } |a - b| = a - b$$

Also, $b - a < 0$.

$$\text{So, } |b - a| = -(b - a) = a - b$$

$$\text{Thus, } |a - b| = |b - a|.$$

$$\text{Use: } |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

In both cases $|a - b| = |b - a|$.



8(b) We break the proof into four cases and use: $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Case 1: Suppose $a \geq 0$ and $b \geq 0$

Then, $ab \geq 0$.

So, $|ab| = ab$, $|a| = a$, and $|b| = b$.

Thus, $|ab| = ab = |a| \cdot |b|$

Case 2: Suppose $a \geq 0$ and $b < 0$

Then, $ab \leq 0$

So, $|ab| = -ab$, $|a| = a$, and $|b| = -b$.

Thus, $|ab| = -ab = a(-b) = |a| \cdot |b|$

Case 3: Suppose $a < 0$ and $b \geq 0$

Then, $ab \leq 0$

So, $|ab| = -ab$, $|a| = -a$, and $|b| = b$.

Thus, $|ab| = -ab = (-a)b = |a| \cdot |b|$

Case 4: Suppose $a < 0$ and $b < 0$

Then, $ab \geq 0$.

So, $|ab| = ab$, $|a| = -a$, and $|b| = -b$.

Thus, $|ab| = ab = (-a)(-b) = |a| \cdot |b|$

In all four cases we get $|ab| = |a| \cdot |b|$.



⑧(c)

Using part (b) we get that

$$\left| \frac{a}{b} \right| = \left| a \cdot \frac{1}{b} \right| = |a| \cdot \left| \frac{1}{b} \right| \quad (*)$$

Use:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

If $b \geq 0$, then $\frac{1}{b} \geq 0$ and $\left| \frac{1}{b} \right| = \frac{1}{b} = \frac{1}{|b|}$

If $b < 0$, then $\frac{1}{b} < 0$ and $\left| \frac{1}{b} \right| = -\left(\frac{1}{b} \right) = \frac{1}{-b} = \frac{1}{|b|}$

Thus, in either case $\left| \frac{1}{b} \right| = \frac{1}{|b|}$

$$\text{Hence, } \left| \frac{a}{b} \right| = |a| \cdot \left| \frac{1}{b} \right| = |a| \cdot \frac{1}{|b|} = \frac{|a|}{|b|}.$$

(*)
from above

(8)(d)

Suppose that $a < x < b$ and $a < y < b$.
We want to show that $|x-y| < b-a$.
We break the proof into two cases.

Case 1: Suppose $x \geq y$.

Then, $x-y \geq 0$.

So, $|x-y| = x-y$.

We know $a < x < b$.

Add $-a$ to get $0 < x-a < b-a$

The equation $a < y$ is given.

So, $-a > -y$.

Thus, $x-a > x-y$

So, $x-y < x-a < b-a$

Hence, $|x-y| = x-y < b-a$.

Use: $|c| = \begin{cases} c & \text{if } c \geq 0 \\ -c & \text{if } c < 0 \end{cases}$

Case 2: Suppose $x < y$.

Then, $x-y < 0$.

So, $|x-y| = -(x-y) = y-x$

We know $a < y < b$.

Add $-a$ to get $0 < y-a < b-a$

The equation $a < x$ is given.

So, $-a > -x$.

Thus, $y-a > y-x$

Hence,

$$y-x < y-a < b-a$$

So,

$$|x-y| = y-x < b-a.$$

In both cases $|x-y| < b-a$.



(8)(e)

Note that

$$|a| = |(a-b) + b| \leq |a-b| + |b|$$

So,

$$|a| - |b| \leq |a-b| \quad (*)$$

triangle inequality

Also,

$$|b| = |(b-a) + a| \leq |b-a| + |a|$$

So,

$$-|b-a| \leq |a| - |b|$$

From part (a) of this problem, $|a-b| = |b-a|$.

Thus, $-|a-b| = -|b-a| \leq |a| - |b| \quad (**)$

Hence, from (*) and (**) we get that

$$-|a-b| \leq |a| - |b| \leq |a-b|$$

Thus,

$$||a| - |b|| \leq |a-b|$$

Here we use this fact from class:
If $c \geq 0$, then
 $|x| \leq c$ iff $-c \leq x \leq c$

